# Optimal Multiple Importance Sampling

SIGGRAPH 2019

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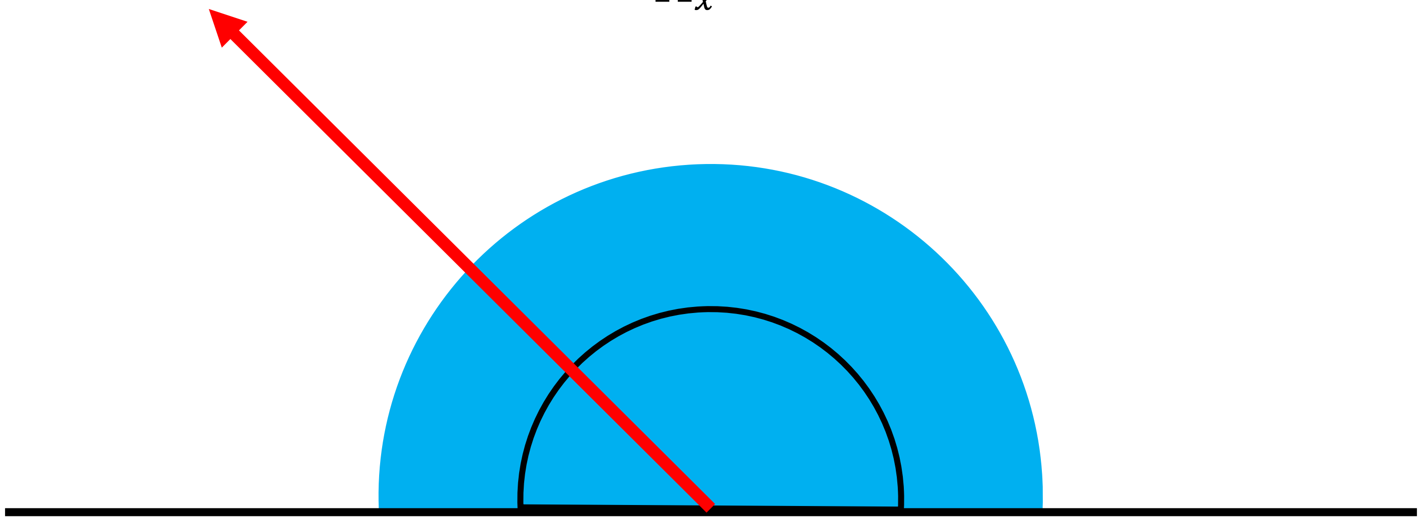
# Previous work

# Previous work – Multiple Importance Sampling

## Rendering Equation

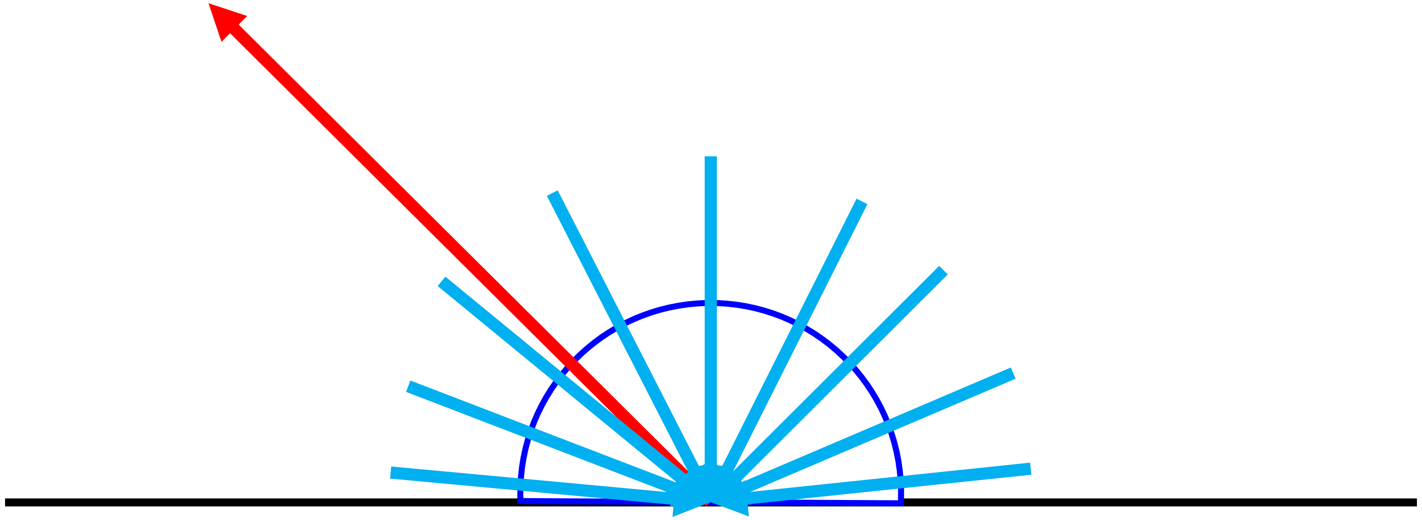
$$L(x \rightarrow \mathbf{\Theta}) += L_e(x \rightarrow \mathbf{\Theta}) + \int_{\Omega_x} f_r(\mathbf{\Psi} \leftarrow \mathbf{\Theta})\, L(x \leftarrow \mathbf{\Psi}) cos\theta_x d\omega_{\mathbf{\Psi}}$$



# Previous work – Multiple Importance Sampling

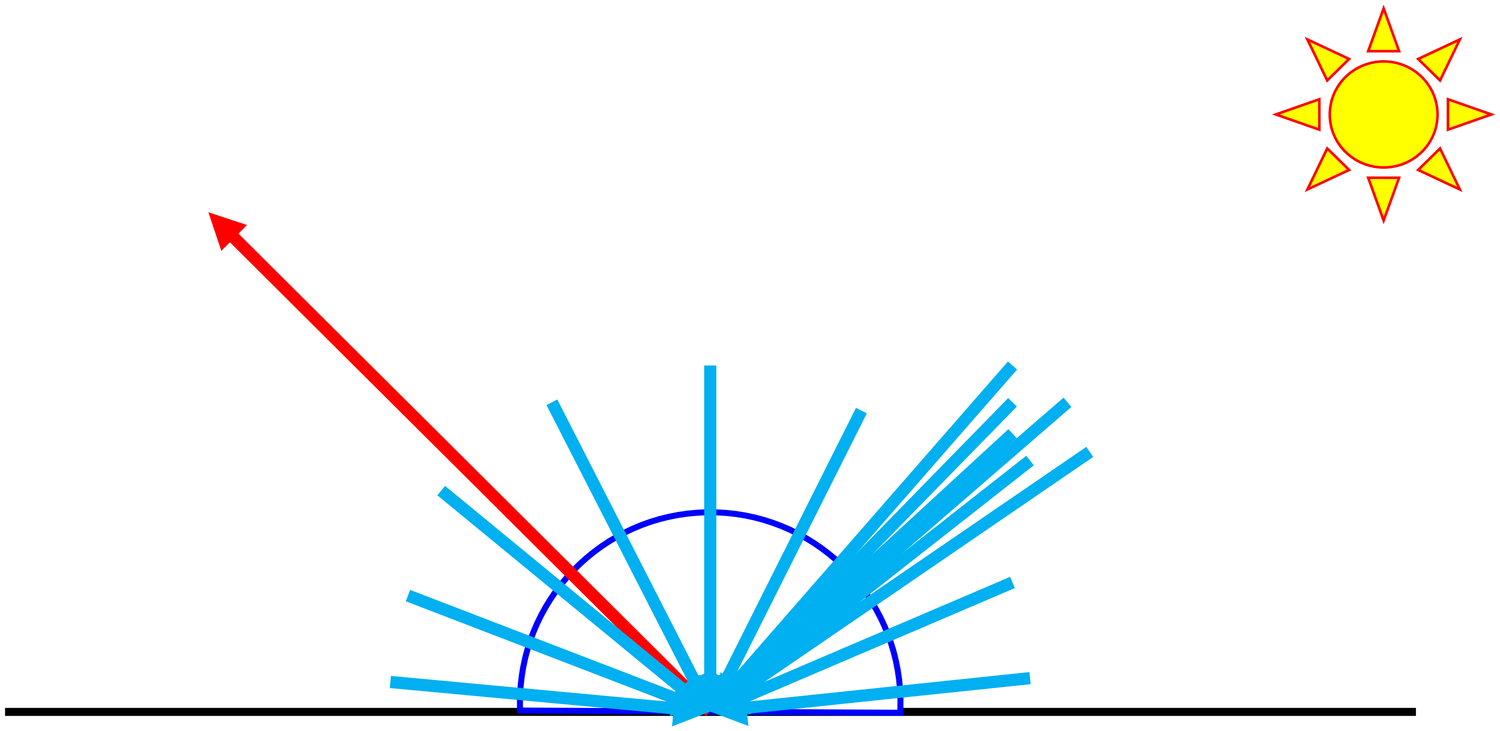
MC estimator

$$F = \int f(x)\, dx \quad \Rightarrow \quad < F > = \frac{1}{N} \sum_i \frac{f(x_i)}{p(x_i)}$$



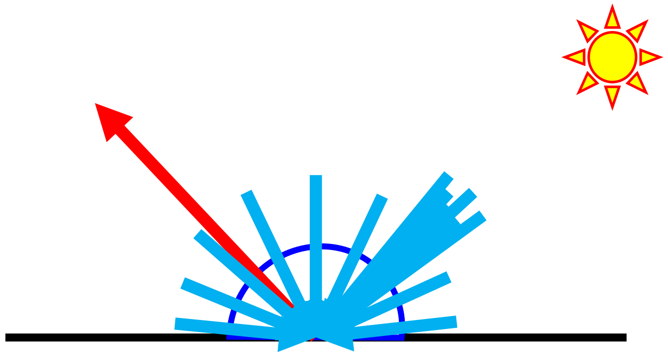
# Previous work – Multiple Importance Sampling

Importance Sampling

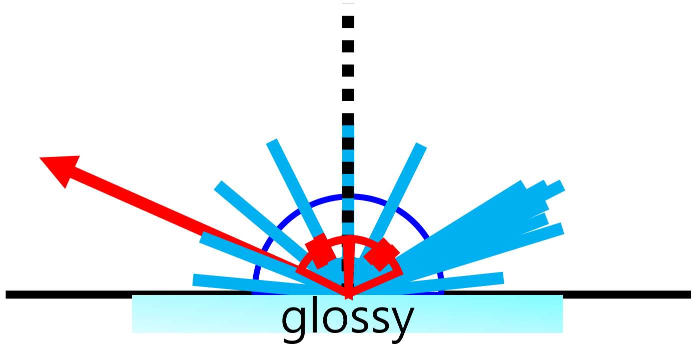


# Previous work – Multiple Importance Sampling

## Multiple Importance Sampling



Light source importance sampling




BRDF importance sampling

$$< F > = \sum_{i=1}^{N} \frac{1}{n_i} \sum_{j=1}^{n_i} \frac{w_i(X_{ij}) f(X_{ij})}{p_i(X_{ij})}$$

# Previous work – Multiple Importance Sampling

## Multiple Importance Sampling

$$< F > = \sum_{i=1}^{N} \frac{1}{n_i} \sum_{j=1}^{n_i} \frac{w_i(X_{ij}) f(X_{ij})}{p_i(X_{ij})}$$

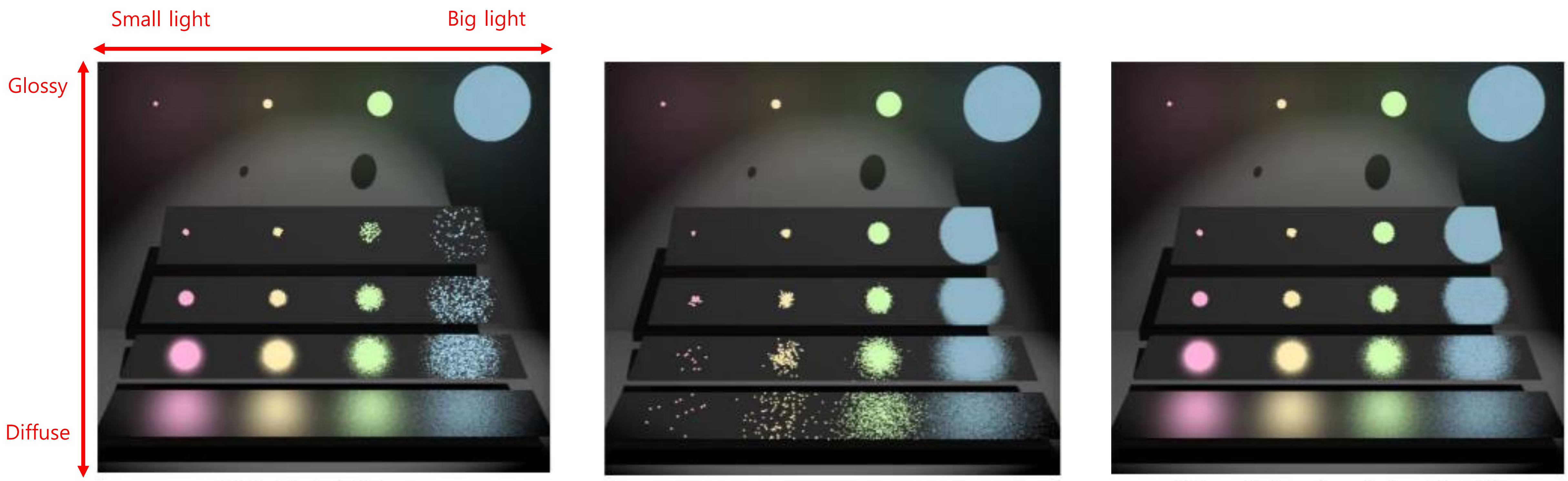
Bounds for unbiased estimator

$$f(x) \neq 0 \Rightarrow \sum_{i=1}^{N} w_i(x) = 1,$$

$$p_i(x) = 0 \Rightarrow w_i(x) = 0,$$

# Previous work – Multiple Importance Sampling

## Multiple Importance Sampling




(a) Sampling the light sources

(b) Sampling the BRDF

(c) A combination of samples from (a) and (b).

*Eric Veach and Leonidas J. Guibas. 1995. Optimally Combining Sampling Techniques for Monte Carlo Rendering. Proc. SIGGRAPH '95.*

# Previous work – Balance and power heuristics

$$w^{p}{}_{i}(x) = \frac{[n_i p_i(x)]^{\beta}}{\sum_{k=1}^{N}[n_k p_k(x)]^{\beta}}$$

- **Balance heuristic**
  - β = 1
  - No other combination strategy can have significantly lower variance than balance heuristic

- **Power heuristic**
  - β > 1
  - Better suited for low-variance problem.

# Previous work – control variates

m : unbiased estimator

t : random variable

E[t] = $\tau$

Let m* = m + c(t – $\tau$)

E(m*) = 0 : m* is unbiased

Var(m*) = Var(m) + $c^2$Var(t) 2cCov(m, t)

When c = $-\frac{\mathrm{Cov(m,t)}}{\mathrm{Var(t)}}$, Var(m*) = $(1-\rho^2_{\mathrm{m,t}})$Var(m) is minimum

Var(m*) = $(1-\rho^2_{\mathrm{m,t}})$Var(m) $\leq$ Var(m)

$\rho_{\mathrm{m,t}} = Corr(m,t)$

# Paper

# Object

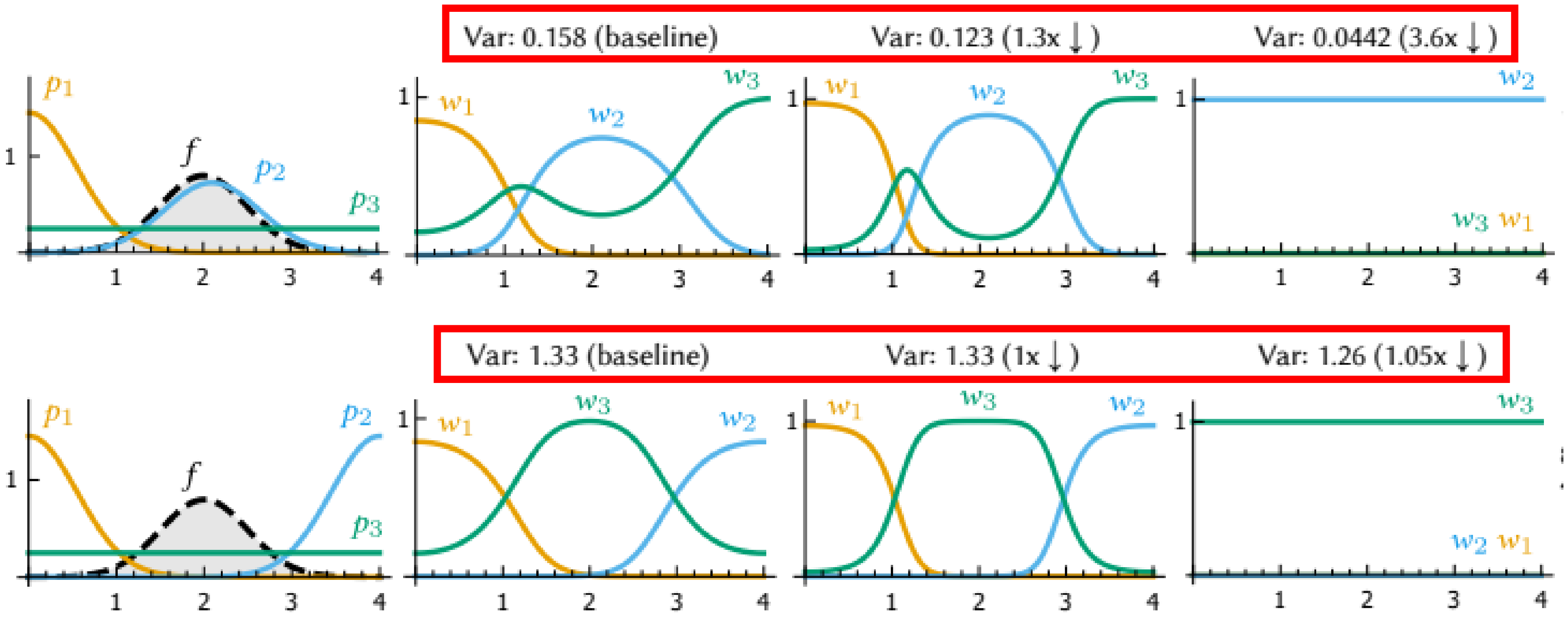
Find optimal weights, $w_i(i=1, \ldots, N)$, that minimize variance of <F>

- Given set of sampling techniques and fixed sampling allocation
- i.e. $p_i$ and $n_i$ ($i = 1, .., N$) are given

$$\langle F\rangle^* = \sum_{i=1}^{N}\sum_{j=1}^{n_i} \frac{w_i(X_{ij}) f(X_{ij})}{n_i p_i(X_{ij})},$$

**Var(<F>)** is minimum

# Motivation




a) integrand and sampling techniques b) balance heuristic weights c) power heuristic weights d) best-technique heuristic weights

**Setting**

1. One-demension
2. Three sampling techniques
3. One sample is taken from each

## One best Importance sampling > MIS !!!

Robustness can lead to low efficiency

# Motivation

Bounds for unbiased estimator

$$f(x) \neq 0 \Rightarrow \sum_{i=1}^{N} w_i(x) = 1,$$
$$p_i(x) = 0 \Rightarrow w_i(x) = 0,$$

*Eric Veach and Leonidas J. Guibas. 1995. Optimally Combining Sampling Techniques for Monte Carlo Rendering. Proc. SIGGRAPH '95.*

$$w_i \geq 0$$

# Motivation

Bounds for unbiased estimator

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*Eric Veach and Leonidas J. Guibas. 1995. Optimally Combining Sampling Techniques for Monte Carlo Rendering. Proc. SIGGRAPH '95.*

+

$$w_i \geq 0$$

*Optimal Multiple Importance Sampling (This paper)*

# Optimal MIS weights

PROBLEM 1. *Given the MIS estimator (1), minimize the functional* $V[w_1, \ldots, w_N] = V[\langle F \rangle^*]$ *in terms of weights* $w_i$*, while maintaining the constraints* $\sum_{i=1}^{N} w_i(x) = 1$ *and* $p_i(x) = 0 \Rightarrow w_i(x) = 0$*, and keeping the number of samples* $n_i$ *and probability densities* $p_i$ *fixed.*

# Optimal MIS weights

THEOREM 5.2. *Let the column vector* $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_N)^\top$ *satisfy the system of linear equations*

$$\mathbf{A}\boldsymbol{\alpha} = \mathbf{b}, \tag{12}$$

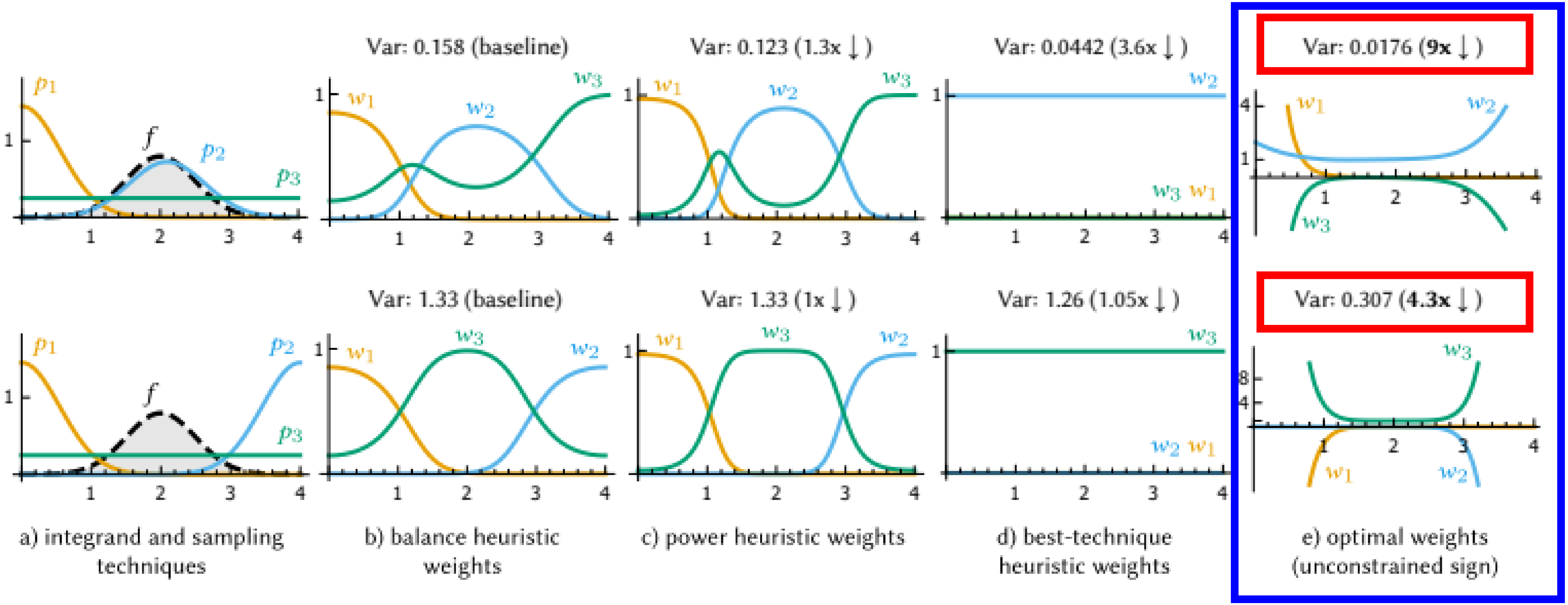
*where* $\mathbf{A}$ *and* $\mathbf{b}$ *are the technique matrix and the contribution vector, respectively. Then the weighting functions*

$$w_i^o(x) = \alpha_i \frac{p_i(x)}{f(x)} + \frac{n_i p_i(x)}{\sum_{j=1}^N n_j p_j(x)} \left(1 - \frac{\sum_{j=1}^N \alpha_j p_j(x)}{f(x)}\right) \tag{13}$$

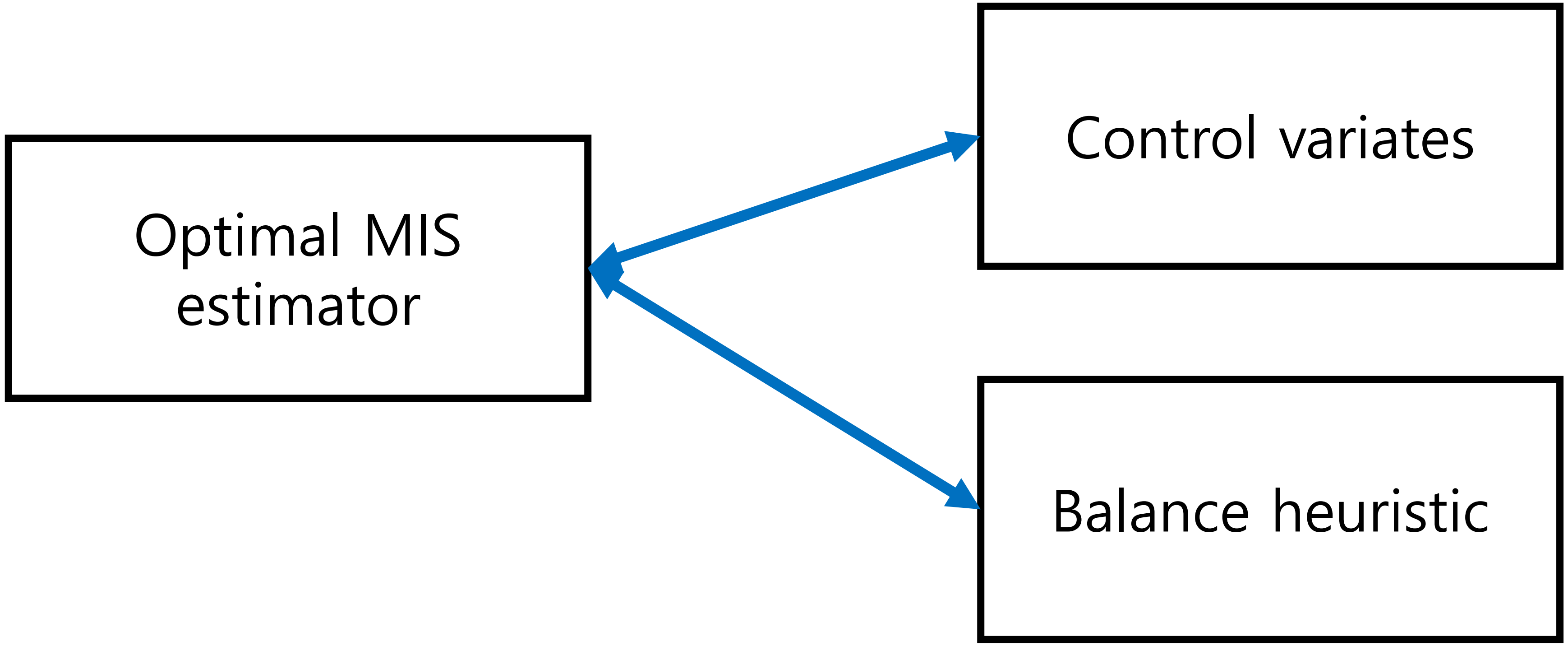
*minimize the functional* $V[w_1, \ldots, w_N]$.

$$\langle F \rangle^o = \sum_{i=1}^N \alpha_i + \frac{1}{M} \sum_{i=1}^N \sum_{j=1}^{n_i} \left( \frac{f(X_{ij})}{p_c(X_{ij})} - \frac{\sum_{k=1}^N \alpha_k p_k(X_{ij})}{p_c(X_{ij})} \right). \tag{16}$$

# Optimal MIS weights



# Theoretical contribution



# Theoretical contribution – control variates

Optimal MIS estimator

$$\langle F \rangle^o = \sum_{i=1}^{N} \alpha_i + \frac{1}{M} \sum_{i=1}^{N} \sum_{j=1}^{n_i} \left( \frac{f(X_{ij})}{p_c(X_{ij})} - \frac{\sum_{k=1}^{N} \alpha_k p_k(X_{ij})}{p_c(X_{ij})} \right). \quad (16)$$

Control variates

$$m^* = m + c(t - \tau), \quad E(t) = \tau$$

$$m = \frac{1}{M} \sum_{i=1}^{N} \sum_{j=1}^{n_i} \left( \frac{f(X_{ij})}{p_c(X_{ij})} \right. \qquad c = -1 \qquad t = \frac{1}{M} \sum_{i=1}^{N} \sum_{j=1}^{n_i} \frac{\sum_{k=1}^{N} \alpha_k p_k(X_{ij})}{p_c(X_{ij})} \qquad \tau = \sum_{i=1}^{N} \alpha_i$$

$$E(t) = \int \sum_{k=1}^{N} a_k p_k(x) dx = \sum_{k=1}^{N} a_k = \tau$$

# Theoretical contribution – relation to balance heuristic

$$Let\ \ g(x) = \sum\nolimits_{k=1}^{N} a_k p_k(x) \quad and \quad p_c(x) = \sum\nolimits_{i=1}^{N} c_i p_i$$

$$\langle F \rangle^o = \sum_{i=1}^{N} \alpha_i + \frac{1}{M} \sum_{i=1}^{N} \sum_{j=1}^{n_i} \left( \frac{f(X_{ij})}{p_c(X_{ij})} - \frac{\sum_{k=1}^{N} \alpha_k p_k(X_{ij})}{p_c(X_{ij})} \right). \quad (16)$$

$$\langle F \rangle = \frac{1}{M} \sum_{i=1}^{N} \sum_{j=1}^{n_i} \frac{f(X_{ij})}{p_c(X_{ij})}, \qquad \langle G_1 \rangle = \frac{1}{M} \sum_{i=1}^{N} \sum_{j=1}^{n_i} \frac{g(X_{ij})}{p_c(X_{ij})}$$

→ Balance heuristic

$$V[\langle F \rangle^o] = V[\langle F \rangle^b] - V[\langle G \rangle^b].$$

→ Optimal MIS variance <= Balance heuristic variance

# Experiment

1. Estimate A and b
2. Estimate $\alpha$ by A$\alpha$= b
3. Estimate <F>by equation 16
   1) Progressive estimator

      Updating A, b, $\alpha$, and F for each iteration

   2) Direct estimator

      Updating A and b for each iteration, and then

      calcuating $\alpha$ and F

THEOREM 5.2. *Let the column vector* $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_N)^\mathsf{T}$ *satisfy the system of linear equations*

$$\mathbf{A}\boldsymbol{\alpha} = \mathbf{b}, \qquad (12)$$

*where* **A** *and* **b** *are the technique matrix and the contribution vector, respectively. Then the weighting functions*

$$w_i^o(x) = \alpha_i \frac{p_i(x)}{f(x)} + \frac{n_i p_i(x)}{\sum_{j=1}^N n_j p_j(x)} \left(1 - \frac{\sum_{j=1}^N \alpha_j p_j(x)}{f(x)}\right) \qquad (13)$$

*minimize the functional* $V[w_1, \ldots, w_N]$.

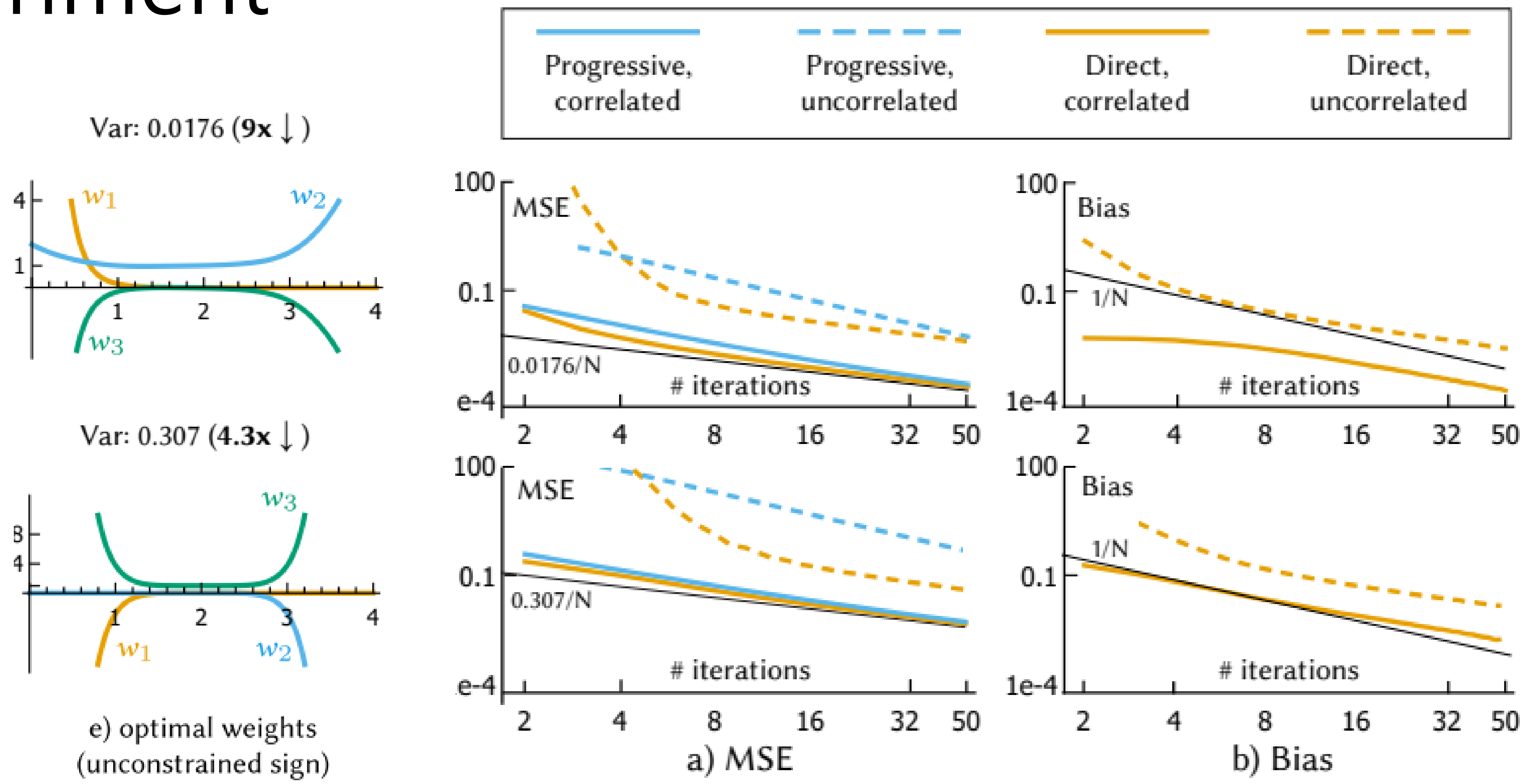
$$\langle F \rangle^o = \sum_{i=1}^N \alpha_i + \frac{1}{M} \sum_{i=1}^N \sum_{j=1}^{n_i} \left( \frac{f(X_{ij})}{p_c(X_{ij})} - \frac{\sum_{k=1}^N \alpha_k p_k(X_{ij})}{p_c(X_{ij})} \right). \qquad (16)$$

$$\langle \mathbf{A} \rangle = \sum_{i=1}^N \sum_{j=1}^{n_i} \mathbf{W}_{ij} \mathbf{W}_{ij}^\mathsf{T}, \quad \langle \mathbf{b} \rangle = \sum_{i=1}^N \sum_{j=1}^{n_i} f(X_{ij})\, S_{ij}\, \mathbf{W}_{ij},$$

$$\mathbf{W}_{ij} = S_{ij} \left(p_1(X_{ij}), \ldots, p_N(X_{ij})\right)^\mathsf{T}$$

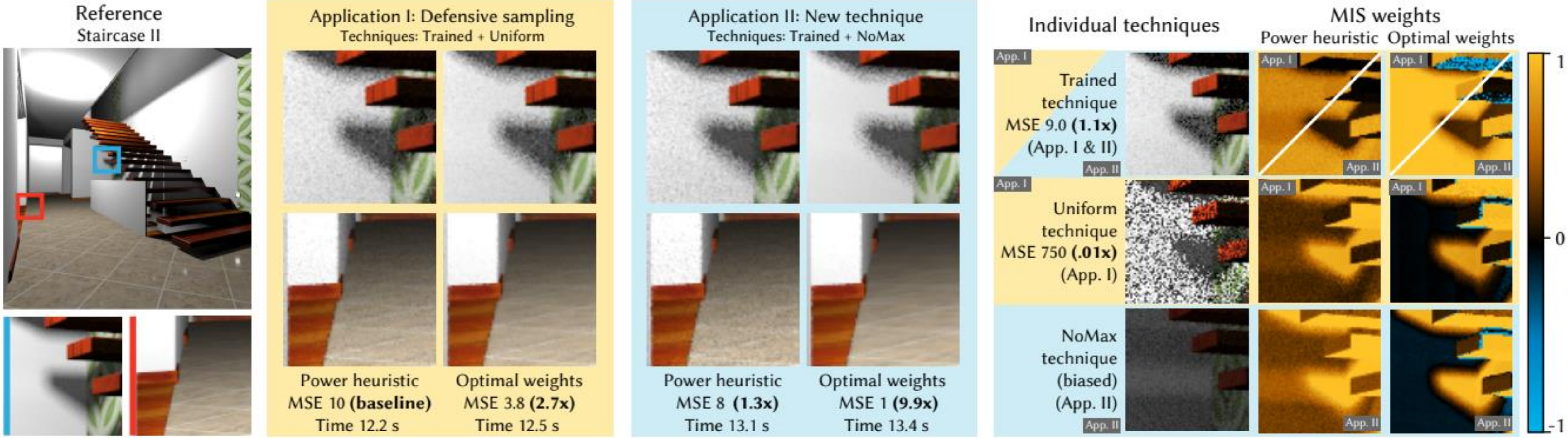
$$S_{ij} = \left(\sum_{k=1}^N n_k p_k(X_{ij})\right)^{-1}$$

# Experiment



Progressive, correlated
Progressive, uncorrelated
Direct, correlated
Direct, uncorrelated
Var: 0.0176 (9x ↓)
w1
w2
w3
Var: 0.307 (4.3x ↓)
MSE
0.0176/N
# iterations
Bias
1/N
0.307/N
100
0.1
e-4
1e-4
2
4
8
16
32
50
e) optimal weights
(unconstrained sign)
a) MSE
b) Bias

# Application



# Thank you